

The fiber homotopically equivalent relation for fiber bundles *Suliman.D and Adem Kılıçman*

Abstract

In this paper, we will give the polyhedron property role to satisfy fiber homotopically equivalent relation in fiber bundle theory over suspensions of polyhedron space.

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1 Introduction

Throughout in this paper the word "space" means Hausdorff space, the word "H-fibration" means an onto regular Hurewicz fibration, and the space of parameterized paths in any space with the compact-open topology. Also we mean by symbol \simeq "homotopy relation". For a path α in any space X , we denote the inverse path of α by $\bar{\alpha}$. In H-fibration $P : E \longrightarrow B$, we denote by $P|_A$ the H-fibration $P|_{P^{-1}(A)} : P^{-1}(A) \longrightarrow A$ defined by $P|_{P^{-1}(A)}(e) = P(e)$ for $e \in P^{-1}(A)$, where A is a subspace of B .

Theorem 1.1 [5] Let X , Y and Z be topological spaces. If X is locally compact and regular space then the map $H : Z \longrightarrow Y^X$ always gives rise to the map $F : Z \times X \longrightarrow Y$ by defining $F(z, x) = H(z)(x)$ for all $x \in X, z \in Z$.

Amin and Alinor [1] introduced the notion of the Sf -function in the theory for fibrations as follows:

Definition 1.2 Let $P : E \longrightarrow B$ be a H-fibration with the a lifting function λ and fiber spaces $F_{b_o} = P^{-1}(b_o)$, where $b_o \in B$. The Sf -function for P induced by λ is a map $\Theta_\lambda : L(B, b_o) \times F_{b_o} \longrightarrow F_{b_o}$ defined by $\Theta_\lambda(\alpha, e) = \lambda(e, \alpha)(1)$ for all $e \in F_{b_o}, \alpha \in L(B, b_o)$, where $L(B, b_o)$ is the set of all loops in B based b_o .

Definition 1.3 Let $P_1 : E_1 \longrightarrow B$ and $P_2 : E_2 \longrightarrow B$ be two H-fibrations with fiber spaces $F_{b_o}^1 = P_1^{-1}(b_o)$ and $F_{b_o}^2 = P_2^{-1}(b_o)$, where $b_o \in B$. The Sf -functions $\Theta_{\lambda_1} : L(B, b_o) \times F_{b_o}^1 \longrightarrow F_{b_o}^1$ and $\Theta_{\lambda_2} : L(B, b_o) \times F_{b_o}^2 \longrightarrow F_{b_o}^2$ are said to be *conjugate* if there is $g \in H(F_{b_o}^1, F_{b_o}^2)$ such that $\Theta_{\lambda_1} \simeq \bar{g} \circ \Theta_{\lambda_2} \circ (id_{L(B, b_o)} \times g)$, where $H(F_{b_o}^1, F_{b_o}^2)$ is the set of all homotopy equivalences from $F_{b_o}^1$ into $F_{b_o}^2$ and \bar{g} denotes the homotopy inverse of g .

Theorem 1.4 [8] Let B be a polyhedron and be the union of two subpolyhedrons B_1 and B_2 such that B_1 is a contractible in B to a point $b_o \in B_1 \cap B_2$ leaves b_o fixed and B_2 is also a contractible to b_o and $B_1 \cap B_2$ be subpolyhedron of B . If $P_1 : E_1 \longrightarrow B$ and $P_2 : E_2 \longrightarrow B$ are two H-fibrations with conjugate Sf -functions Θ_{λ_1} and Θ_{λ_2} by $g \in H(F_{b_o}^1, F_{b_o}^2)$, then P_1 and P_2 are fiber homotopy equivalent.

Definition 1.5 Let E , B , and F be spaces. Let $P : E \longrightarrow B$ be a map of E onto B and G be group of all homeomorphisms of F onto F with as a binary usual composition operation \circ . Then $\gamma = (E, P, B, F, G)$ is said to be a *fiber bundle over a base B* if there is an open covering $\{V_j : j \in \Lambda\}$ of B , where Λ is index set and for each $j \in \Lambda$, there is a homeomorphism $\theta_j : V_j \times F \longrightarrow P^{-1}(V_j)$ (called coordinate function) such that:

- 1- $P[\theta_j(b, y)] = b$ for all $b \in V_j, y \in F$.
- 2- For each pair $i, j \in \Lambda$ and $b \in V_i \cap V_j$, the homeomorphism $\theta_{jb}^{-1} \circ \theta_{ib} : F \longrightarrow F$ corresponds to an element of G , where $\theta_{kb} : F \longrightarrow P^{-1}(b)$ defined by $\theta_{kb}(y) = \theta_k(b, y)$ for all $b \in V_k, y \in F$ and $k = i, j$.
- 3- For each pair $i, j \in \Lambda$, the function $g_{ij} : V_i \cap V_j \longrightarrow G$ given by $g_{ij}(b) = \theta_{jb}^{-1} \circ \theta_{ib}$ is continuous (called coordinate transformation).

In fiber bundle $\gamma = (E, P, B, F, G)$, we shall denote the identity element of a group G by e , the inverse element $g \in G$ by g^{-1} , we mean by a map $k : (X, x_o) \longrightarrow (Y, y_o)$ a map k of X into Y and $k(x_o) = y_o$.

It's clear that every fiber bundle $\gamma = (E, P, B, F, G)$ is a regular local Hurewicz fibration and if B is paracompact then γ is Hurewicz fibration.

Theorem 1.6 [6] Let S^n be the n -sphere in R^{n+1} , where $n > 0$ is a positive integer. Then for a fiber bundle $\gamma = (E, P, S^n, F, G)$, there is a characteristic map $\mu : (S^{n-1}, x_o) \longrightarrow (G, e)$.

Now we recall the Dold's theorem in fiber bundles over sphere S^n as follows:

Theorem 1.7 [The Dold's theorem]

Let $\gamma = (E, P, S^n, F, G)$ and $\gamma' = (E', P', S^n, F', G')$ be two fiber bundles over sphere S^n with locally compact fibers F and F' . Let $\mu : (S^{n-1}, x_o) \longrightarrow (G, e)$ and $\mu' : (S^{n-1}, x_o) \longrightarrow (G', e')$ be characteristic maps of γ and γ' , respectively and let $i : G \longrightarrow H(F, F)$ and $i' : G' \longrightarrow H(F', F')$ be the inclusion maps. Then γ and γ' are fiber homotopy equivalent if and only if there is homotopy equivalence $g : F \longrightarrow F'$ such that the maps

$$q(x) = g \circ (i \circ \mu)(x) \circ \bar{g} \quad \text{and} \quad q'(x) = (i' \circ \mu')(x)$$

of S^{n-1} into $H(F', F')$ are homotopic.

Definition 1.8 Let X be any space and x_o be a base point in X . The *suspension* $S(X)$ of X is defined to be the quotient space of $X \times I$ in which for all $x \in X$, $(x, 0)$ identified to $(x_o, 0)$, $(x, 1)$ identified to $(x_o, 1)$, and $X \times \{1/2\}$ identified to X .

The Dold's theorem remains valid if we use suspensions of polyhedron spaces instead of n -spheres S^n for the base of bundles, [2].

Theorem 1.9 [6] The suspension $S(S^n)$ of n -sphere S^n is homeomorphic to sphere S^{n+1} , where $n > 0$ is a positive integer.

Theorem 1.10 [2] If X is a polyhedron space, then $S(X)$ is also polyhedron space.

2 Sf -function and fiber bundles

In this paper, we will show the Sf -function role to satisfy fiber homotopically equivalent relation in fiber bundle theory, in particular, we will the equivalently between our theorem(1.4) and Dold's theorem over suspensions of polyhedron spaces.

As mentioned perviously that Dold's theorem remains valid if we use suspensions of polyhedron space instead of n -spheres S^n for the base of bundles. Hence theorem (1.6) also remains valid with suspensions of polyhedron spaces. That is, for a fiber bundle $\gamma = (E, f, S(X), F, G)$ over suspension $S(X)$ of polyhedron space X , there is a characteristic map $\mu : (X, x_o) \longrightarrow (G, e)$. In the following theorem, we will prove that the converse of this theorem is also true for suspension $S(X)$ of any space X .

Theorem 2.1 Let G be a group of all homeomorphisms of space F with as binary usual composition operation \circ and X be any space. If there is a map $\mu : (X, x_o) \longrightarrow (G, e)$, then there is bundle E over $S(X)$ and a map $P : E \longrightarrow S(X)$ such that $\gamma = (E, P, S(X), F, G)$ is fiber bundle.

Proof. By the definition of $S(X)$, we can consider $S(X)$ as the union of two cones, one of them is $C_0(X) = X \times [0, 1/2]$ with $(x, 0)$ identified to $(x_o, 0)$ and the other $C_1(X) = X \times [1/2, 1]$ with $(x, 1)$ identified to $(x_o, 1)$. Let $V_1 = C_0(X)$ and $V_2 = C_1(X)$, then $S(X) = V_1 \cup V_2$ and there is a retraction $r : V_1 \cap V_2 \longrightarrow X$. Now define maps

$$g_{ii} : V_i \longrightarrow G, \quad g_{ii}(x) = e \quad \forall x \in V_i, (i = 1, 2),$$

$$g_{12} : V_1 \cap V_2 \longrightarrow G, \quad g_{12}(x) = (\mu \circ r)(x) \quad \forall x \in V_1 \cap V_2,$$

and

$$g_{21} : V_1 \cap V_2 \longrightarrow G, \quad g_{21}(x) = [g_{12}(x)]^{-1} \quad \forall x \in V_1 \cap V_2.$$

Let $J = \{1, 2\}$ be a space with the discrete topology and let $T \subset S(X) \times F \times J$ be the set defined by

$$T = \{(x, y, j) : x \in V_j, y \in F, j \in J\}.$$

Define an equivalent relation \equiv on T by

$$(x_1, y_1, j) \equiv (x_2, y_2, k) \iff x_1 = x_2 \quad \text{and} \quad g_{kj}(x_1)(y_1) = y_2,$$

where $(x_1, y_1, j), (x_2, y_2, k) \in T$. Then put E be the set of equivalence classes so obtained with the quotient topology. Hence define a map $P : E \longrightarrow S(X)$ by

$$P([(x, y, j)]) = x \quad \forall [(x, y, j)] \in E,$$

and the maps $\theta_j : V_j \times F \longrightarrow P^{-1}(V_j)$ defined by

$$\theta_j(x, y) = [(x, y, j)] \quad \forall (x, y) \in V_j \times F.$$

Hence it's clear that $\gamma = (E, P, S(X), F, G)$ is fiber bundle. \diamond

Theorem 2.2 Let $P : E \longrightarrow B$ be H-fibration with locally compact fiber $F = P^{-1}(b_o)$, where $b_o \in B$. Then the function $\phi : L(B, b_o) \longrightarrow F^F$ given by

$$\phi(w)(x) = \Theta_\lambda(w, x) \quad \forall w \in L(B, b_o), x \in F,$$

is a continuous function of $L(B, b_o)$ into $H(F, F)$.

Proof. Since F is a locally compact and Hausdorff, then F is regular. Hence by theorem (1.1), the function ϕ is continuous. Now we will prove that for $w \in L(B, b_o)$, $\phi(w)$ is homotopy equivalence from F into F . For $w \in L(B, b_o)$, we can define a map $\overline{\phi(w)} : F \longrightarrow F$ by

$$\overline{\phi(w)}(x) = \Theta_\lambda(\overline{w}, x) \quad \forall x \in F.$$

Then we get that

$$[\phi(w) \circ \overline{\phi(w)}](x) = \lambda[\lambda(x, \overline{w})(1), w](1) \quad \forall x \in F,$$

and

$$\overline{\phi(w)} \circ \phi(w)(x) = \lambda[\lambda(x, w)(1), \overline{w}](1) \quad \forall x \in F.$$

Then by lemma (2.2) in [8],

$$\phi(w) \circ \overline{\phi(w)} \simeq i_F \quad \text{and} \quad \overline{\phi(w)} \circ \phi(w) \simeq i_F.$$

Hence $\phi(w) : F \longrightarrow F$ is homotopy equivalence, that is, $\phi(w) \in H(F, F)$. Therefore ϕ is a map from $L(B, b_o)$ into $H(F, F)$. \diamond

Theorem 2.3 Let $\gamma = (E, P, B, F, G)$ be a fiber bundle admits a lifting function λ with locally compact fiber F . Then the function $\phi : L(B, b_o) \longrightarrow F^F$ given by

$$\phi(w)(x) = \Theta_\lambda(w, x) \quad \forall w \in L(B, b_o), x \in F,$$

is a map from $L(B, b_o)$ into G .

Proof. Since F is a locally compact, then by theorem (1.1), the function ϕ is continuous. For $w \in B^I$, let $F_{w(0)} = P^{-1}(w(0))$ and let $F_{w(1)} = P^{-1}(w(1))$. Then the map $A : F_{w(0)} \longrightarrow F_{w(1)}$ given by

$$A(x) = \lambda(x, w)(1) \quad \forall x \in F_{w(0)},$$

is homeomorphism since it is obtained from the compositions of coordinate functions which are homeomorphisms. Hence ϕ is a map from $L(B, b_o)$ into G . \diamond

To prove the equivalently between theorem (1.4) and Dold's theorem, we first have to rephrase theorem (1.4) for two H-fibrations over a common suspension base as follow:

For any space E with fixed point $x_o \in E$, there is a *conical map* $\psi : E \longrightarrow L(S(E), x_o)$ gives as follows:

consider $S(E)$ as the union of two cones one of them $C_0(E) = E \times [0, 1/2]$ with $(x, 0)$ identified to $(x_o, 0)$ and the other $C_1(E) = E \times [1/2, 1]$ with $(x, 1)$ identified to $(x_o, 1)$. $C_0(E)$ can be contracted on itself to x_o leaving x_o fixed and similarly for $C_1(E)$. And for $x \in E$, let $w_0(x)$ be path between x and x_o in $C_0(E)$ and let $w_1(x)$ be path between x and x_o in $C_1(E)$. Define the *conical map* $\psi : E \longrightarrow L(S(E), x_o)$ by

$$\psi(x) = \overline{w_1(x)} \star w_0(x) \quad \forall x \in E.$$

Let $\gamma = (E, P, S(X), F, G)$ and $\gamma' = (E', P', S(X), F', G')$ be two H-fibrations over a common suspension base $S(X)$ of a polyhedron space X with with locally compact fibers F and F' . In Figure 6.1, let

$$\mu : (X, x_o) \longrightarrow (G, e) \quad \text{and} \quad \mu' : (X, x_o) \longrightarrow (G', e')$$

be characteristic maps of γ and γ' , respectively. And

$$i : G \longrightarrow H(F, F) \quad \text{and} \quad i' : G' \longrightarrow H(F', F')$$

be the inclusion maps. From theorems (2.2) and (2.3), then theorem (1.4) and Dold's theorem can now be compared.

$$\begin{array}{ccccc}
 & & L(S(X), x_o) & & \\
 & \swarrow \phi' & \uparrow \psi & \searrow \phi & \\
 G' & \xleftarrow{\mu'} & X & \xrightarrow{\mu} & G \\
 \downarrow i' & & & & \downarrow i \\
 H(F', F') & \xleftarrow{T(f)=g \circ f \circ [\bar{g}] \quad \forall f \in H(F, F)} & & & H(F, F)
 \end{array}$$

Where $g \in H(F, F')$ and ψ is the conical map.

Hence theorem (1.4) can then be restated in terms of ϕ , ϕ' , and ψ as follows: Two H-fibrations $\gamma = (E, P, S(X), F, G)$ and $\gamma' = (E', P', S(X), F', G')$ are fiber homotopy equivalent if and only if there is $g \in H(F, F')$ such that two maps

$$m(x) = g \circ i \circ \phi[\psi(x)] \circ \bar{g} \quad \forall x \in X,$$

and

$$m'(x) = i' \circ \phi'[\psi(x)] \quad \forall x \in X,$$

from X into $H(F', F')$ are homotopic.

Now if $\phi \circ \psi \simeq \mu$ and $\phi' \circ \psi \simeq \mu'$, then theorem (1.4) and Dold's theorem are equivalent. In the following theorem, we will prove the desired.

Theorem 2.4 Let $\gamma = (E, P, S(X), F, G)$ be a fiber bundle over suspension $S(X)$ of a polyhedron space X with locally compact fiber F and admits a lifting function λ . And let $\phi : L(S(X).x_o) \longrightarrow G$ be a map given by

$$\phi(\beta)(x) = \Theta_\lambda(\beta, x) \quad \forall \beta \in L(S(X), x_o), x \in F.$$

Then $\phi \circ \psi \simeq \mu$, where $\mu : (X, x_o) \longrightarrow (G, e)$ is the characteristic map of γ and ψ is the conical map.

Proof. Let $B = S(X) = C_0(X) \cup C_1(X)$, $B_1 = C_0(X)$, and $B_2 = C_1(X)$. It's clear that $x_o \in X = B_1 \cap B_2$. Now define maps

$$g_{ii} : B_i \longrightarrow G, \quad g_{ii}(x) = e \quad \forall x \in B_i, (i = 1, 2),$$

$$g_{12} : X \longrightarrow G, \quad g_{12}(x) = \mu(x) \quad \forall x \in X,$$

and

$$g_{21} : X \longrightarrow G, \quad g_{21}(x) = [\mu(x)]^{-1} \quad \forall x \in X.$$

Let $J = \{1, 2\}$ be a space with the discrete topology and let $T \subset S(X) \times F \times J$ be the set defined by

$$T = \{(x, y, j) : x \in B_j, y \in F, j \in J\}.$$

Define an equivalent relation \equiv on T by

$$(x_1, y_1, j) \equiv (x_2, y_2, k) \iff x_1 = x_2 \quad \text{and} \quad g_{kj}(x_1)(y_1) = y_2,$$

where $(x_1, y_1, j), (x_2, y_2, k) \in T$.

Recall theorem (2.1) that points of E are identified to the equivalent classes of all trips $(x, y, j) \in T$. Hence for $j = 1, 2$, the maps $\epsilon_j : B_j \times F \longrightarrow P^{-1}(B_j)$ given by

$$\epsilon_j(x, y) = [(x, y, j)] \quad \forall (x, y) \in B_j \times F, \quad (1)$$

are fiber homeomorphisms which denote the equivalence class of the triple (x, y, j) . Hence put $y = [(x_o, y, j)]$, where $j = 1, 2$.

Hence we can define lifting functions λ_1 and λ_2 for fibrations $P|_{B_1}$ and $P|_{B_2}$, respectively, as follow:

$$\Delta^1 P = \{(e, w) \in E \times B_1^I : P(e) = w(0)\},$$

$$\Delta^2 P = \{(e, w) \in E \times B_2^I : P(e) = w(0)\},$$

$$\overline{\Delta} P = \{(\beta, x) \in L(B, b_o) \times F : \beta = w_2 \star w_1,$$

where $w_i \in B_i^I (i = 1, 2)$ and $\beta(1/2) = w_2(1) = w_1(0) \in B_1 \cap B_2\}$,

$$\lambda_1(e, w) = \epsilon_1[w(t), (\pi_2 \circ \epsilon_1^{-1})(e)] \quad \forall (e, w) \in \Delta^1 P, \quad (2)$$

and

$$\lambda_2(e, w) = \epsilon_2[w(t), (\pi_2 \circ \epsilon_2^{-1})(e)] \quad \forall (e, w) \in \Delta^2 P, \quad (3)$$

where π_2 is the second projection.

Since λ is lifting function for γ , then it is also lifting function for $P|B_1$ and $P|B_2$. Hence $\lambda \simeq \lambda_1$ on $\Delta^1 P$ and $\lambda \simeq \lambda_2$ on $\Delta^2 P$. Hence the map $\bar{\phi}: \bar{\Delta} P \rightarrow F$ given by

$$\bar{\phi}(\beta, x) = \lambda_1[\lambda_2(x, w_2)(1), w_1](1) \quad \forall (\beta, x) \in \bar{\Delta} P, \quad (4)$$

is homotopic to the map $\tilde{\phi}: \bar{\Delta} P \rightarrow F$ defined by

$$\tilde{\phi}(\beta, x) = \lambda[\lambda(x, w_2)(1), w_1](1) \quad \forall (\beta, x) \in \bar{\Delta} P. \quad (5)$$

That is, $\tilde{\phi} \simeq \bar{\phi}$ and by lemma (2.2) in [8], $\tilde{\phi} \simeq \Theta_\lambda$. Hence

$$\Theta_\lambda \simeq \bar{\phi}. \quad (6)$$

Now from the equations 1,2,3, we have that for $\beta \in L(S(X), x_o)$,

$$\beta = w_2 \star w_1, \quad w_1 \in B_1^I, w_2 \in B_2^I,$$

$$\lambda_1(y, w_1)(1) = [(w_1(1), y, 1)],$$

and

$$\begin{aligned} \lambda_2(y, w_2)(1) &= [(w_2(1), y, 2)] \\ &= [(w_2(1), \mu(w_2(1))(y), 1)]. \end{aligned}$$

Hence

$$\begin{aligned} \bar{\phi}(\beta, x) &= \lambda_1[\lambda_2(x, w_2)(1), w_1](1) \\ &= [(x_o, \mu(w_2(1))(y), 1)] \\ &= \mu(w_2(1))(y) \\ &= \mu(\beta(1/2))(y). \end{aligned}$$

Let $\bar{L}(S(X), x_o)$ be the projection of $\bar{\Delta} P$ on $L(S(X), x_o)$ and $\bar{\bar{\phi}}: \bar{L}(S(X), x_o) \rightarrow G$ be a map given by

$$\bar{\bar{\phi}}(\beta)(x) = \bar{\phi}(\beta, x) \quad \forall \beta \in \bar{L}(S(X), x_o), x \in F.$$

Then by the equation 6, $\phi \simeq \bar{\bar{\phi}}$ and

$$(\bar{\bar{\phi}} \circ \psi)(x) = \bar{\bar{\phi}}[\psi(x)] = \mu[\psi(x)(1/2)] = \mu(x),$$

that is, $\bar{\bar{\phi}} \circ \psi = \mu$. Hence $\phi \circ \psi \simeq \mu$. Therefore theorem (1.4) and the Dold's theorem are equivalent. \diamond

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